# Support of generalized parton distributions in Bethe–Salpeter models of hadrons

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## Abstract

The proper support of generalized parton distributions from relativistic constituent quark models with pointlike constituents is studied. The correct support is guaranteed when the vertex function does not depend on the relative minus-momentum. We show that including quark interactions in models with pointlike constituent quarks might lead to a support problem. A computation of the magnitude of the support problem in the Bonn relativistic constituent quark model is presented.
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Historically, form factors and parton distribution functions have played a key role in hadron physics. In 1997, a new research tool emerged in the framework of the generalized parton distributions (GPDs) [1], providing us with a natural unification of form factors and parton distribution functions [2]. GPDs describe the soft, i.e. non-perturbative hadronic, scattering part of Hard Exclusive Meson Production (HEMP) and Deeply Virtual Compton Scattering (DVCS). Today, a lot of experimental and theoretical effort is devoted to the study of these generalized parton distributions, of which is hoped that they will shed light on the origin of the spin of hadrons in general, and the nucleon in particular.

At present, calculating GPDs from first QCD principles is not feasible, and one relies on chiral perturbation theory [3], lattice QCD [4], parametrizations [5] or phenomenological models [6–16]. This Letter focuses on the latter class. In order to compare the GPDs computed in a phenomenological model
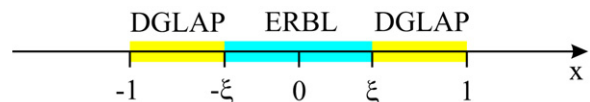


Fig. 1. Pictorial indication of the supported regions.

with the DVCS and HEMP data, a $Q^2$-evolution needs to be performed [17]. The evolution equations depend highly on the kinematic region of the process. More specifically, one can distinguish between the DGLAP and ERBL regions, where the $Q^2$-evolution is respectively governed by the DGLAP and the ERBL equations. As indicated schematically in Fig. 1, parton model constraints make GPDs vanish outside these two regions [18,19]. A model is said to have the correct support, when it can resolve the DGLAP and ERBL regions, and has a vanishing GPD outside this region. Developing such a model is far from straightforward. In this Letter we will show how the support can arise naturally in quark models that are based on the Bethe–Salpeter approach. Including quark dynamics in the model, however, is likely to destroy the correct support.

In the following, we will focus on the GPD of a pseudoscalar meson such as the pion, considering the case where the partons do not transfer helicity. Pseudoscalar mesons have only

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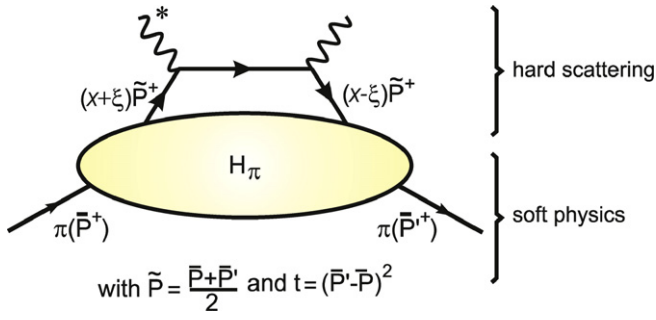[1] Current address: Universidad Complutense de Madrid, Ciudad Universitaria s/n, E-28040 Madrid.

Fig. 2. Feynman diagram of DVCS, with definition of the kinematical variables $x$, $\xi$ and $t$.

one such generalized quark distribution associated with them, whereas the nucleon has four. The theoretical analysis of the support region is, therefore, less cumbersome for the pion GPD than for the nucleon ones.

GPDs are non-diagonal matrix elements of a bilocal field operator on the light cone. For partons with spin 1/2 (quarks) in a pseudoscalar meson, the GPD associated with helicity conserving partons is defined as follows [18]:

$$
H_\pi(x,\xi,t)
$$
$$
= \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ix\tilde{P}^+ z^-} \langle \bar{P}' | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \psi\left(\frac{z}{2}\right) | \bar{P} \rangle \bigg|_{z^+=0, \mathbf{z}_\perp = \mathbf{0}}.
\tag{1}
$$

In this equation, $\tilde{P} = \frac{\bar{P}'+\bar{P}}{2}$. Definition (1) uses light-cone coordinates. A four-momentum $p = (p^0, p^1, p^2, p^3)$ can be written in light-cone coordinates as $p = [p^+, p^-, \mathbf{p}_\perp]$ with $p^\pm = (p^0 \pm p^3)/\sqrt{2}$ and $\mathbf{p}_\perp = (p^1, p^2)$.

In order to calculate the GPD, several choices regarding the kinematics of the problem have to be made. In fact, two choices are used in literature [20,21]. We make use of the definitions from [20], illustrated in Fig. 2 for the DVCS process: $x$ denotes the fraction of the average pion plus-momentum that is reabsorbed by the meson, while the skewedness $\xi$ is a measure for the plus-momentum that is lost in the process:

$$
\xi = \frac{\bar{P}^+ - \bar{P}'^+}{\bar{P}^+ + \bar{P}'^+}.
\tag{2}
$$

$\xi$ is defined on the interval $[0, \xi_{max}]$ with

$$
\xi_{max} = \sqrt{\frac{-t}{4M_\pi^2 - t}},
\tag{3}
$$

where $t = \Delta^2 = (\bar{P}' - \bar{P})^2$. The above-mentioned DGLAP and ERBL regions are defined as follows:

- $x \in [-1, -\xi]$: DGLAP region (emission and absorption of antiquark);
- $x \in [-\xi, \xi]$: ERBL region (emission of both quark and antiquark);
- $x \in [\xi, 1]$: DGLAP region (emission and absorption of quark).

This is depicted schematically in Fig. 1. As mentioned before, the GPD should vanish outside these regions, i.e. for $|x| > 1$.

In a Bethe–Salpeter quark model, the bilocal current matrix element from Eq. (1) can be written in terms of the Bethe–Salpeter amplitudes $\chi_{\bar{P}}$:

$$
\langle \bar{P}' | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \psi\left(\frac{z}{2}\right) | \bar{P} \rangle
$$
$$
= \int d^4 x_2'\, i \, \text{Tr}\left\{ \bar{\chi}_{\bar{P}'}\left(x_2', \frac{z}{2}\right) \left( (i\slashed{\partial}_{x_2'} - m_1) \chi_{\bar{P}}\left(x_2', -\frac{z}{2}\right) \right) \gamma^+ \right\}
$$
$$
+ \int d^4 y_1'\, i \, \text{Tr}\left\{ \left( (i\slashed{\partial}_{y_1'} - m_2) \bar{\chi}_{\bar{P}'}\left(-\frac{z}{2}, y_1'\right) \right) \gamma^+ \chi_{\bar{P}}\left(\frac{z}{2}, y_1'\right) \right\}.
\tag{4}
$$

The first (second) term on the right-hand side of this equation refers to the coupling of the bilocal current to the antiquark (quark). Both terms can be treated in exactly the same manner. In the remainder of this Letter, we focus on the analysis of the second term of Eq. (4). Inserting the quark term in Eq. (1) and making use of the definition of the quark and antiquark propagators,

$$
(i\slashed{\partial}_x - m) S^F(x) = i\delta^{(4)}(x - x'),
\tag{5}
$$

we can rewrite the quark GPD as an integral in momentum space:

$$
H_\pi^q(x,\xi,t)
$$
$$
= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \delta\left( \frac{2x + \xi - 1}{2(1+\xi)} \bar{P}^+ - p^+ \right)
$$
$$
\times \text{Tr}\left\{ \bar{\Gamma}_{\bar{P}'}\left(p + \frac{\Delta}{2}\right) S_1^F\left(\frac{\bar{P}'}{2} + p + \frac{\Delta}{2}\right) \gamma^+ \right.
$$
$$
\times \left. S_1^F\left(\frac{\bar{P}}{2} + p\right) \Gamma_{\bar{P}}(p) S_2^F\left(-\frac{\bar{P}}{2} + p\right) \right\}.
\tag{6}
$$

On the left-hand side of Eq. (6), the superscript $q$ indicates that only the coupling to the quark is considered. The vertex functions

$$
\Gamma_{\bar{P}} = \left(S_1^F\right)^{-1} \chi_{\bar{P}} \left(S_2^F\right)^{-1}
\tag{7}
$$

have been introduced on the right-hand side of the equation, with subscript 1 (2) referring to the quark (antiquark). The pictorial representation of Eq. (6) is sketched in Fig. 3. In some Bethe–Salpeter models, see e.g. Ref. [22], the quark and antiquark propagators are assumed to have the form of a free propagator of pointlike fermions, with a constituent mass $m$:

$$
S_{1,2}^F(p) = i \frac{\slashed{p} + m_{1,2}}{p^2 - m_{1,2}^2 + i\epsilon}.
\tag{8}
$$

As it appears, the denominators of the three propagator terms in Eq. (6) suggest immediately the correct support region for the GPDs. Since the poles of the propagators are Lorentz invariants, any convenient frame can be chosen to prove this statement. The GPD in Eq. (1) was defined for a coordinate system with $\vec{\tilde{P}}$ along the $z$-axis [1]. The Breit frame is such a coordinate system, with $\bar{P} = (\bar{M}, -\frac{\vec{\Delta}}{2})$ and $\bar{P}' = (\bar{M}, \frac{\vec{\Delta}}{2})$. Here, $\bar{M}^2 = M^2 - \frac{t}{4}$ and $\vec{\Delta} = (\vec{\Delta}_\perp, -2\xi\bar{M})$ with $\xi$ as introduced before. We emphasize that these expressions uniquely apply to the calculation of the *elastic* GPD, $M' = M$.
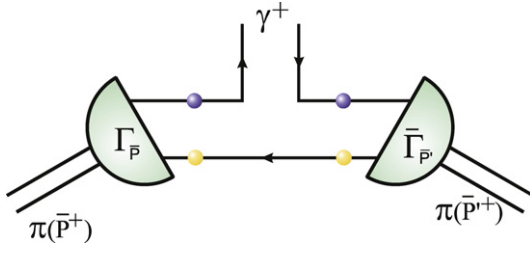
Fig. 3. Schematic representation of the expression of the pion quark GPD in terms of Bethe–Salpeter vertex functions (Eq. (6)).



Fig. 4. The pion GPD $H_{\pi^+}^u$ as a function of $x$, calculated in the Bonn constituent quark model at $t = -0.1$ GeV$^2$. Results are shown for $\xi = 0$ (solid line), $\xi = 0.3$ (dashed line) and $\xi = 0.7$ (dotted line).

Making use of the $\delta$-function to perform the integration over $p^+$, it is now possible to write the propagator denominators as:

$$\left(\left(\frac{\bar{P}}{2} + p + \Delta\right)^2 - m_1^2 + i\epsilon\right) = \mathcal{A} + \sqrt{2}\bar{M}(x - \xi)p^- + i\epsilon,$$

$$\left(\left(\frac{\bar{P}}{2} + p\right)^2 - m_1^2 + i\epsilon\right) = \mathcal{B} + \sqrt{2}\bar{M}(x + \xi)p^- + i\epsilon,$$

$$\left(\left(-\frac{\bar{P}}{2} + p\right)^2 - m_2^2 + i\epsilon\right) = \mathcal{C} + \sqrt{2}\bar{M}(x - 1)p^- + i\epsilon,$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ do not depend on $p^-$. From these expressions, it is clear that the propagators have poles in the complex $p^-$-plane. The position of these poles with respect to the real $p^-$-axis (upper or lower half-plane) depends on the kinematics, more specifically on the sign of $(x - \xi)$, $(x + \xi)$ and $(x - 1)$. For $x \notin [-\xi, 1]$, the propagator poles lie on the same side of the real axis. Furthermore, the propagators are analytic in all other points of the complex plane. Assuming that the vertex function $\Gamma$ does not contain poles in the complex $p^-$-plane, and making use of Cauchy's theorem, one can conclude that the momentum dependence of the propagators ensures the correct support for the GPDs.

The above considerations are clearly valid if the vertex function $\Gamma$ and its adjoint do not depend on $p^-$ and are therefore free from poles when analytically continued in the complex $p^-$-plane. The calculation of the pion GPD with constant vertex functions has been performed by S. Noguera et al. in the NJL-model [10] with a Pauli–Villars regularization procedure. The calculation of GPDs with $p^-$-independent vertex functions was performed by H.-M. Choi et al. [12,13], as well as by B.C. Tiburzi and G.A. Miller [14–16]. These authors made use of a reduction of Bethe–Salpeter amplitudes to light-cone wave functions by projection on the light-cone. In Refs. [14–16], the scalar Wick–Cutkosky model was adopted and successfully applied to the calculation of scalar meson GPDs. In Refs. [12,13], the light-front Bethe–Salpeter vertex functions were replaced by wave functions obtained in a light-front constituent quark model. All of these calculations yielded the correct support, in agreement with the present analysis.

In instant-form models, however, the vertex functions can depend on $p^-$, which makes them have a different pole structure. In this case, the correct support can no longer be guaranteed. To prove this, we start with Liouville's theorem, which states that *the only bounded entire functions are the constant*
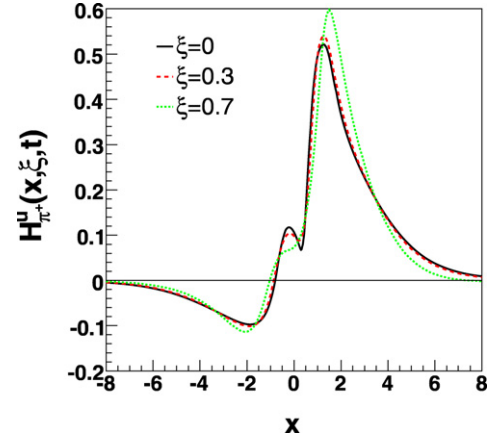
*functions*. As a result, a vertex function which depends on the momentum variable $p^-$ will not be bounded for all points in the complex $p^-$-plane. Moreover, the radial part of a vertex function for a ground-state particle is a real-valued function on the real $p^-$-axis, up to a constant phase factor. This is an important constraint. Indeed, for a holomorphic function $f$ whose restriction to the real numbers is real-valued it can be proven that $f(p^{-*}) = f^*(p^-)$. Here, $*$ stands for complex conjugation. In other words: if $f$ has a singularity in the upper half-plane, it also has one in the lower half-plane, and vice versa. Combining these two statements, we find that a dynamic, momentum dependent vertex function will either have complex conjugated poles, or a singularity on both complex half-circles at $\Re(p^-) \pm i\infty$. In the latter case, Cauchy's theorem is no longer applicable, while in the former case, there will always be at least one pole in each half-plane. Consequently, it is not a priori clear whether the GPD will vanish outside the interval $x \in [-1, 1]$. This means that the correct support can no longer be guaranteed. Similar arguments apply to relativistic quark–diquark models of the nucleon with pointlike constituents.

We stress that this does not mean that there will necessarily be a support problem in Bethe–Salpeter constituent quark models with pointlike constituents containing dynamics—it merely shows that a support problem can *no longer be excluded*. The above considerations indicate that guaranteeing the correct support puts non-trivial constraints on the analytic properties of the Bethe–Salpeter vertex functions. Whether these are fulfilled in particular models has to be checked in each case individually. As an example, we show in Fig. 4 the pion GPD calculated in the framework of the covariant quark model of the Bonn group [22].

In the Bonn model, the interaction kernels are instantaneous, i.e. independent from relative energy variables in the meson rest frame. More specifically, the employed interquark interactions are the linearly rising confinement interaction and the 't Hooft instanton induced interaction [23]. The confinement interaction has an appropriate Dirac structure to optimize the splitting between spin–orbit partners. The 't Hooft instanton induced in-

teraction acts as a residual interaction and accounts for the mass splittings in the pseudoscalar and scalar sectors [24]. Together with the assumption of free fermion propagators with an effective mass (Eq. (8)), the instantaneous approximation allows to reduce the full (but unsolvable) Bethe–Salpeter equation to a solvable Salpeter equation [25]. The resulting vertex function is independent of the relative energy variable, but does depend on the relative minus-momentum. The Bonn model has only seven parameters which are fitted to the static properties of the meson spectrum. In Refs. [22,25–27], the model is described in detail.

Fig. 4 shows the result of a calculation of the pion up-quark GPD $H_{\pi+}^u$ as a function of $x$ at $t = -0.1$ GeV$^2$ and three different values for $\xi$ ($\xi = 0$, $\xi = 0.3$ and $\xi = 0.7$). The first moment of the GPD,

$$\int_{-\infty}^{\infty} dx \, H_{\pi+}(x, \xi, t) \tag{9}$$

yields the electromagnetic form factor, while the isospin symmetry relation [18],

$$H_{\pi+}^u(x, \xi, t) = -H_{\pi+}^d(-x, \xi, t), \tag{10}$$

is exactly fulfilled. Here, $H_{\pi+}^d$ is calculated from the first term on the right-hand side of Eq. (4), which describes the coupling of the bilocal current to the antiquark. As a measure of the support problem, the support parameter $\phi$ can be introduced:

$$\phi = \frac{\int_{-\xi}^{1} |H_{\pi+}^u(x, \xi, t)| \, dx}{\int_{-\infty}^{\infty} |H_{\pi+}^u(x, \xi, t)| \, dx}. \tag{11}$$

When the appropriate kinematical regions are resolved (see Fig. 1), this fraction equals one. In contrast, $\phi < 1$ implies that the GPD is non-zero outside the supported region, which is $[-\xi, 1]$ for the quark term. The smaller the fraction, the worse the support. In the case of Fig. 4, this fraction equals $\phi = 0.13$ for $\xi = 0$ and $\xi = 0.3$. For $\xi = 0.7$, one has $\phi = 0.12$. From these numbers, it is clear that the correct support is violated in the Bonn constituent quark model.

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